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A FRACTIONAL INTEGRAL OPERATOR CORRESPONDING TO NEGATIVE POWERS OF A CERTAIN SECOND ORDER DIFFERENTIAL OPERATOR

Preprint

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 26A33, 33A30, 44A15.

A fractional integral operator corresponding to negative powers of a certain second order differential operator *)

by

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ABSTRACT

A class of integral operators is defined which contains negative fractional powers of both $d^2/dx^2 + \frac{\nu}{x}\frac{d}{dx}$ and $\frac{1}{x}\frac{d}{dx}$. These operators are intimately connected with an integral formula for hypergeometric functions due to Erdélyi. A result of Wimp on hypergeometric integral equations is also contained in the theory of these fractional integral operators.

KEY WORDS & PHRASES: *fractional calculus; hypergeometric integral transfroms.*

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*) This paper is not for review; it is meant for publication elsewhere.

1. INTRODUCTION AND PRELIMINARIES

In this paper a class of integral operators is studied which contains negative fractional powers of the second order operator

$$D_\nu = \frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx} .$$

The fractional integrals of the Riemann-Liouville type corresponding to (d/dx) are well-known and they are widely used in applied mathematics and in the theory of special functions. The theory of the Riemann-Liouville integral can be found in "The Fractional Calculus" by OLDHAM & SPANIER [9], in the proceedings of the conference in New Haven (1974), edited by ROSS [11], and in the references given in these works.

The operator D_ν is also important, and for applications it seems natural to introduce a fractional calculus for this operator, similar to that for the Riemann-Liouville integral. The operator $I_\nu^{\mu,\lambda}$ introduced in this paper contains negative powers of both D_ν and x^{-1} d/dx. The attractiveness of combining the operators D_ν and x^{-1} d/dx arises from their simple commutation relation (cf. (1.10)).

Let us give a summary of the contents. At the end of this section a number of results on hypergeometric functions is given. In section 2 we solve some I.V.P.'s (initial value problems) in the operators x^{-1} d/dx and D_ν. In order to solve these problems, we consider for $\nu \in \mathbb{N}$ the operators D_ν as the "radial" part of the wave operator in $\mathbb{R}^{\nu+1}$. The fractionalization of the wave operator is given by the Riesz distributions Z_μ (for references, see section 2). By the integration of Z_μ over the inessential variables, an integral operator is found which has the desired properties (now it is no longer necessary to restrict ν to $\mathbb{N}$ and one can take $\nu \in \mathbb{C}$). Once the integral operator is found, we will give "classical" proofs for the theorems in section 2. The solutions of the I.V.P.'s in section 2 serve as a basis for the introduction of the fractional integral operator $I_\nu^{\mu,\lambda}$. This operator is defined in section 3, where we also study the composition rule and some other properties of $I_\nu^{\mu,\lambda}$. A further analysis of the operators $I_\nu^{\mu,\lambda}$ and their connection with the Erdélyi-Kober operators are given in section 6, but first we discuss some applications in sections 4 and 5. In section 4

Erdélyi's integral formula (1.11) is interpreted as a fractional integral. In section 5 it is shown how $I_{\nu}^{\mu,\lambda}$ and its inverse lead to a pair of integral equations, which, after some transformations, yield one of the results of WIMP [16].

Finally sections 7 and 8 deal with distributions. Section 7 summarizes a few properties of the distributions and test functions needed in section 8. In section 8 it is shown how the operator $I_{\nu}^{\mu,\lambda}$ can be extended to an operator acting on a certain class of distributions. To define the action of $I_{\nu}^{\mu,\lambda}$ we will use a certain procedure which Erdélyi [3] called the "method of adjoints".

Throughout this paper we will refer to the following results on hypergeometric functions, which are defined by

$$(1.1) \qquad {}_pF_q(a_1,a_2,\ldots,a_p;b_1,\ldots,b_q;x) := \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k k!} x^k,$$

$$|x| < 1, \; c \neq 0,-1,-2,\ldots, p \leq q+1$$

cf. ERDÉLYI et al. [5, Chapter II]. Two special cases of (1.1) are

$$(1.2) \qquad {}_2F_1(a,b;b;x) = {}_1F_0(a;\;;x) = (1-x)^{-a},$$

and

$$(1.3) \qquad {}_2F_1(0,b;c;x) = 1.$$

It can be readily verified by termwise differentiation of (1.1) that the following differentiation formulas hold.

$$(1.4) \qquad \left(-\frac{1}{x}\frac{d}{dx}\right)\frac{(1-x^2)^{c-1}}{2^c\Gamma(c)}\,{}_2F_1(a,b;c;1-x^2) =$$

$$= \frac{(1-x^2)^{c-2}}{2^{c-1}\Gamma(c-1)}\,{}_2F_1(a,b;c-1;1-x^2),$$

and

$$(1.5)\quad \left(\frac{d^2}{dx^2}+\frac{2(a+b-c)+1}{x}\frac{d}{dx}\right)\frac{(1-x^2)^{c-1}}{2^c\Gamma(c)}\;{}_2F_1(a,b;c;1-x^2) = \frac{(1-x^2)^{c-3}}{2^{c-2}\Gamma(c-2)}\;{}_2F_1(a-1,b-1;c-2;1-x^2)$$

Formula (1.4) is similar to [5, 2.8 (22)]. Formula (1.5) is due to Tom Koornwinder. It is closely connected with the identity

$$(1.6)\quad \left(\frac{d^2}{dx^2}+\frac{2c-1}{x}\frac{d}{dx}\right)\frac{(1-x^2)^{a+b-c+2}}{2^{a+b-c+2}\Gamma(a+2-c)\Gamma(b+2-c)}\;{}_2F_1(a+1,b+1;c;x^2) = \frac{(1-x^2)^{a+b-c}}{2^{a+b-c}\Gamma(a-c+1)\Gamma(b-c+1)}\;{}_2F_1(a,b;c;x^2),$$

which is proved and applied in KOORNWINDER [7]; By substitution in (1.6) of

$$(1.7)\quad {}_2F_1(a,b;c;x^2) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\;{}_2F_1(a,b;a+b+1-c;1-x^2) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\;(1-x^2)^{c-a-b}\;{}_2F_1(c-a,c-b;c+1-a-b;1-x^2)$$

(cf. [5, 2.9 (33)]), we see that this formula splits into (1.5) and

$$(1.8)\quad \left(\frac{d^2}{dx^2}+\frac{2(a+b-c)+1}{x}\frac{d}{dx}\right)\;{}_2F_1(a,b;c;1-x^2) = \frac{4ab(a-c)(b-c)}{c(c+1)}\;{}_2F_1(a+1,b+1;c+2;1-x^2).$$

Iteration of (1.4) and (1.5) yields:

$$\left(-\frac{1}{x}\frac{d}{dx}\right)^{\ell}\left(\frac{d^2}{dx^2}+\frac{2(a+b-c)+1}{x}\frac{d}{dx}\right)^{k}\frac{(1-x^2)^{c-1}}{2^c\Gamma(c)}\;{}_2F_1(a,b;c;1-x^2) = \frac{(1-x^2)^{c-\ell-2k-1}}{2^{c-\ell-2k}\Gamma(c-\ell-2k)}\;{}_2F_1(a-k,b-k;c-\ell-2k;1-x^2).$$

An important role is played by the commutation relation of the operators in (1.9):

$$(1.10)\qquad (-\frac{1}{x}\frac{d}{dx})(\frac{d^2}{dx^2}+\frac{\nu}{x}\frac{d}{dx}) = (\frac{d^2}{dx^2}+\frac{\nu+2}{x}\frac{d}{dx})(-\frac{1}{x}\frac{d}{dx}).$$

ERDÉLYI [4] derived the following important formula by means of fractional differentiation by parts:

$$(1.11)\qquad {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)}\int_0^1 w^{s-1}(1-w)^{c-s-1}(1-wz)^{r-a-b}\,. \; {}_2F_1(r-a,r-b;s;wz)\,{}_2F_1(a+b-r,r-s;c-s;\frac{(1-w)z}{1-wz})dw,$$

$\mathrm{Re}\ c > \mathrm{Re}\ s > 0$.

2. SOME INITIAL VALUE PROBLEMS

The formulas obtained in this section form the basis for the definition of the fractional integral operator to be discussed later, but they are also of interest by themselves. The first initial value problem (lemma 2.1) is essentially contained in the theory of the Riemann-Liouville integral. The second one (theorem 2.2), however, seems to be new, although the elements for its solution are available in literature (see also section 6).

We will start with the following I.V.P., which is simply solved by using x^2 as a new variable in the Riemann-Liouville integral.

<u>LEMMA 2.1</u>. *Let* $g \in C((0,1])$, *then the unique solution of*

$$(2.1)\qquad \begin{cases} (-\frac{1}{x}\frac{d}{dx})^{\ell} f(x) = g(x), & 0 < x \le 1, \\ f^{(i)}(1) = 0, & i = 0,1,\dots,\ell-1, \\ f \in C^{\ell}((0,1]), & \end{cases}$$

is given by

$$(2.2) \qquad f(x) = \frac{1}{\Gamma(\ell)} \int_x^1 y\left(\frac{y^2-x^2}{2}\right)^{\ell-1} g(y)\,dy.$$

THEOREM 2.2. *Let* $g \in C((0,1])$, *then the unique solution of*

$$(2.3) \qquad \begin{cases} \left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^k f(x) = g(x), & 0 < x \le 1,\ \nu \in \mathbb{C}, \\ f^{(i)}(1) = 0, & i = 0,1,\ldots,2k-1, \\ f \in C^{2k}((0,1]), & \end{cases}$$

is given by

$$(2.4) \qquad f(x) = \frac{1}{\Gamma(2k)} \int_x^1 \left(\frac{y^2-x^2}{2y}\right)^{2k-1} {}_2F_1\left(k + \frac{\nu-1}{2}, k; 2k; 1 - \frac{x^2}{y^2}\right) g(y)\,dy.$$

PROOF. We will use the adhoc notation

$$\int_x^1 M_k(x,y)g(y)\,dy$$

for the right hand side of (2.4). For $k > 1$ we have

$$\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right) \int_x^1 M_k(x,y)g(y)\,dy =$$

$$\int_x^1 \left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right) M_k(x,y)g(y)\,dy =$$

$$\int_x^1 M_{k-1}(x,y)g(y)\,dy.$$

The first step is correct because $M_k(x,x) = 0$. The second step is an application of (1.5). Iteration yields

$$\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^{k-1} f(x) = \int_x^1 \left(\frac{y^2-x^2}{2y}\right) {}_2F_1\left(\frac{\nu+1}{2}, 1; 2; 1 - \frac{x^2}{y^2}\right) g(y)\,dy,$$

Applying the operator again and using (1.4) and (1.2), we get

$$\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^k f(x) =$$

$$= \left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right) \int_x^1 \left(\frac{y^2-x^2}{2y}\right) {}_2F_1\left(\frac{\nu+1}{2},1;2;1-\frac{x^2}{y^2}\right) g(y)\,dy =$$

$$= \left(\frac{d}{dx} + \frac{\nu}{x}\right) \int_x^1 \frac{d}{dx}\left[\frac{y^2-x^2}{2y}\ {}_2F_1\left(\frac{\nu+1}{2},1;2;1-\frac{x^2}{y^2}\right)\right] g(y)\,dy =$$

$$= -\left(\frac{d}{dx} + \frac{\nu}{x}\right) \int_x^1 \frac{x}{y}\ {}_2F_1\left(\frac{\nu+1}{2},1;1;1-\frac{x^2}{y^2}\right) g(y)\,dy =$$

$$= -\left(\frac{d}{dx} + \frac{\nu}{x}\right) \int_x^1 \left(\frac{y}{x}\right)^\nu g(y)\,dy =$$

$$= \left(\frac{y}{x}\right)^\nu g(y) \Big|_{y=x} - \int_x^1 \left(\frac{d}{dx} + \frac{\nu}{x}\right) \left(\frac{y}{x}\right)^\nu g(y)\,dy$$

$$= g(x).$$

Thus (2.4) is a solution of (2.3). The initial conditions are fulfilled because of the factor $(x^2-y^2)^{2k-1}$ in the integrand. The uniqueness of the solution results from the standard theory for I.V.P.'s. □

In the proof of theorem 2.2 it was checked that the solution (2.4) indeed satisfies (2.3). Let us now indicate how (2.4) was obtained. For $\nu = 1,2,\ldots$ the operator $d^2/dx^2 + \nu x^{-1}\ d/dx$ is that part of the wave operator in $\mathbb{R}^{\nu+1}$ which depends only on the Lorentz distance $x = (x_0^2 - x_1^2 - \ldots - x_\nu^2)^{1/2}$ in $\mathbb{R}^{\nu+1}$. To the function g on $\mathbb{R}^+$ we let correspond the function $\tilde{g}$ in the backward light cone defined by

$$\tilde{g}(x_0,\ldots,x_\nu) = g\left(\sqrt{x_0^2 - x_1^2 - \ldots - x_\nu^2}\right) = g(x)$$

The I.V.P. (2.3) corresponds in $\mathbb{R}^{\nu+1}$ to the problem

$$\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \ldots - \frac{\partial^2}{\partial x_\nu^2}\right)^k \tilde{f}(x_0,x_1,\ldots,x_\nu) = \Box^k \tilde{f}(x_0,x_1,\ldots,x_\nu) = \tag{2.5}$$

$$= \tilde{g}(x_0,x_1,\ldots,x_\nu),$$

for the iterated wave operator, with initial conditions for the derivatives in the direction of the Lorentz normal on the sheet $x_0 < 0$ of the hyperbola $x_0^2 - x_1^2 - \ldots - x_\nu^2 = 1$. We are interested in the solution at the right hand side of this sheet (i.e. forwards in time). So let us consider $\tilde{f}$ and $\tilde{g}$ as distributions in the backward light cone with supports in the region bounded to the left by the left hand sheet of the hyperbola $x_0^2 - x_1^2 - \ldots - x_\nu^2 = 1$. Then (2.5) holds in distributional sense in the backward light cone. The unique solution $\tilde{f}$ of (2.3) is given in terms of the Riesz distribution Z_{2k} as

(2.6) $$\tilde{f} = Z_{2k} * \tilde{g},$$

where the asterisk denotes the convolution product in distributional sense, which can be interpreted in the classical way, if Z_{2k} and $\tilde{g}$ are regular distributions (locally integrable functions) and if the (Lebesque) integral defining the convolution product converges absolutely. Here Z_{2k} is given by

(2.7) $$Z_\mu(x_0, \ldots, x_\nu) := [\pi^{\frac{\nu-1}{2}} 2^{\mu-1} \Gamma(\tfrac{\mu}{2}) \Gamma(\tfrac{\mu-\nu+1}{2})]^{-1} \rho^{\mu-\nu-1},$$

$$\rho = \begin{cases} \sqrt{x_0^2 - x_1^2 - \ldots - x_\nu^2} & \text{if } x_0 \geq \sqrt{x_1^2 + \ldots + x_\nu^2}, \\ 0 & \text{elsewhere,} \end{cases}$$

for $\mathbb{R}e\ \mu > \nu-1$, and by

$$\langle Z_\mu, \phi\rangle = \langle Z_{\mu+2k}, \Box^k \phi\rangle$$

for $\mathbb{R}e\ \mu > \nu-2k-1$ $(k \in \mathbb{N})$, where ϕ is a C^∞-function on $\mathbb{R}^{\nu+1}$ with compact support. The Riesz distributions have the following properties

$$Z_\mu * Z_\nu = Z_{\mu+\nu},$$

$$Z_0 = \delta$$

$$Z_{-2k} = \Box^k \delta.$$

They have their supports in the forward light cone. This distribution Z_μ is an entire function of μ in a weak sense, i.e., (Z_μ,ϕ) is an entire function of μ for each C^∞-function ϕ with compact support. It is a regular distribution for $\mathrm{Re}\ \mu > \nu-1$. A good introduction to distribution theory and to the solution of (2.5) in terms of Riesz distributions is given by DE JAGER [6]. He restricts himself, however, to I.V.P.'s for (2.5) with initial conditions on the plane $x_0 = 0$. For the more general case with initial conditions on some space oriented hyperplane we refer to RIESZ [10].

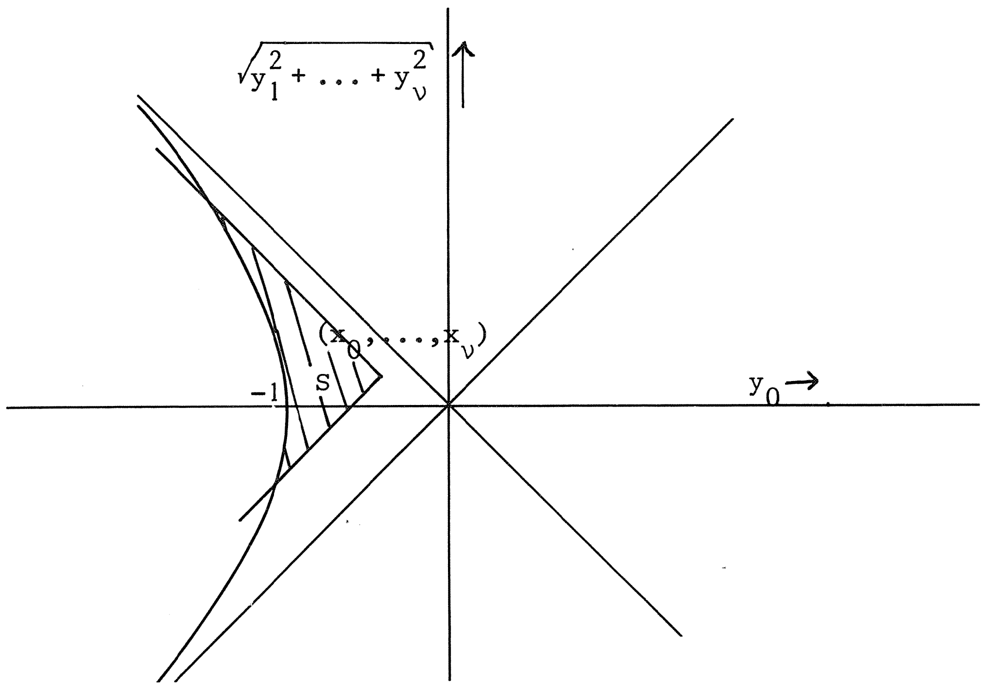

Figure 1

In the following we will replace 2k in (2.6) by μ, which may be arbitrary complex. The restriction of the function $\tilde{f}$ to the interior of the backward light cone is a sufficient (and, in the case of Lorentz invariance, necessary) condition for the existence of the convolution in (2.6), because in this case the intersection of the supports of $\tilde{g}(y_0,y_1,\ldots,y_\nu)$ and $Z_\mu(x_0-y_0,\ldots,x_\nu-y_\nu)$ is bounded. In the backward light cone the generalized solution $\tilde{f}$ as given in (2.6) can be written as a classical integral if $\mathrm{Re}\ \mu > \nu-1$. See figure 1 for the region S of integration, which contributes to the convolution integral (2.6). If $\tilde{g}$ is invariant under Lorentz transformations, then $\tilde{f}$ is invariant as well. Hence, a solution $\tilde{f}$ of (2.5) results in a solution f of the original problem (2.3):

$$f(x) = \tilde{f}(x_0,\dots,x_\nu), \qquad x = \sqrt{x_0^2 - x_1^2 - \dots - x_\nu^2}, \qquad 0 < x \le 1.$$

Consider

$$\tilde{f}(x_0,\dots,x_\nu) = \int_S \tilde{g}(y_0,\dots,y_\nu) Z_\mu(x_0-y_0,\dots,x_\nu-y_\nu)\,dy_0\dots dy_\nu,$$

where $S = \{(y_0,\dots,y_\nu) \in \mathbb{R}^{\nu+1} \mid y_0^2 - y_1^2 - \dots - y_\nu^2 \le 1,\ (x_0-y_0)^2 - (x_1-y_1)^2 - \dots - (x_\nu-y_\nu)^2 \ge 0,\ y_0 < x_0 < 0\}$ (see figure 1). Instead of $\tilde{f}(x_0,\dots,x_\nu)$ and $\tilde{g}(y_0,\dots,y_\nu)$ we will shortly write f(x) and g(y). By S is denoted both the region in $\mathbb{R}^{\nu+1}$ as defined above and the corresponding region after a change of variables. Because of the Lorentz symmetry we can take $(x_0,x_1,\dots,x_\nu) = (-x,0,\dots,0)$, $0 < x \le 1$. Let us make the transformation of variables $y_0 = -y \cosh t$, $y_1 = y \sinh t\ \omega_1,\dots,y_\nu = y \sinh t\ \omega_\nu$, with $\omega_1,\dots,\omega_\nu$ coordinates on the unit sphere in $\mathbb{R}^\nu$. This yields

$$f(x) = [\pi^{\frac{\nu-1}{2}} 2^{\mu-1}\Gamma(\tfrac{\mu}{2})\Gamma(\tfrac{\mu-\nu+1}{2})]^{-1}\,.$$

$$\cdot\int_S g(y)\{(y \cosh t - x)^2 - y^2 \sinh t\}^{\frac{\mu-\nu-1}{2}} y^\nu (\sinh t)^{\nu-1} dy\,dt\,d\Omega_\nu$$

$$= \frac{1}{\Gamma(\mu)} \int_{y=x}^{1} \left(\frac{y^2-x^2}{2y}\right)^{\mu-1} {}_2F_1\left(\frac{\mu+\nu-1}{2},\frac{\mu}{2};\mu;1-\frac{x^2}{y^2}\right) g(y)\,dy.$$

Here $d\Omega_\nu$ denotes integration over the unit sphere in $\mathbb{R}^\nu$, and $S = \{(y,t) \mid y > x > 0,\ t > 0,\ (y \cosh t - x)^2 - y^2 \sinh^2 t > 0\}$. We used the transformations of variables:

$$\sinh^2 \frac{t}{2} = \psi$$

and

$$\psi = \left[\frac{4xy}{(y-x)^2}(\chi+1)\right]^{-1},$$

and the formulas

$$\Omega_\nu = \frac{2\pi^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})},$$

$$\int_0^\infty s^{b-1}(1+s)^{a-c}(1+sz)^{-a}ds = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;1-z),$$

$${}_2F_1(a,b;2b;\frac{4z}{(1+z)^2}) = (1+z)^{2a}\,{}_2F_1(a,a-b+\tfrac{1}{2};b+\tfrac{1}{2};z^2),$$

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z),$$

and the duplication formula for the gamma function

$$\Gamma(2z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{2}).$$

For these formulas, one is referred to ERDÉLYI et al. [5]. In this way we have found (2.4) for $\nu = 1,2,3,\ldots$ and $\mathbb{R}\mathrm{e}\ \mu > \nu-1$. The restriction on μ can be immediately removed because Z_μ is an analytic function of μ. After the solution in the form (2.4) has been obtained for $\nu = 1,2,\ldots$, the general case can be proved, as described earlier.

Combination of lemma 2.1 and theorem 2.2 leads to

THEOREM 2.3. *Let* $g \in C((0,1])$, *then the unique solution of the I.V.P.*

$$(2.8)\qquad \begin{cases} \left(-\frac{1}{x}\frac{d}{dx}\right)^{\ell}\left(\frac{d^2}{dx^2}+\frac{\nu}{x}\frac{d}{dx}\right)^k f(x) = g(x), & 0 < x \le 1, \\ f^{(i)}(1) = 0, & i = 0,1,2,\ldots,\ell+2k-1, \\ f \in C^{\ell+2k}((0,1]), \end{cases}$$

is given by

$$(2.9)\qquad f(x) = \frac{1}{\Gamma(\ell+2k)} \int_x^1 \left(\frac{y^2-x^2}{2}\right)^{\ell+2k-1} y^{1-2k}$$

$$ {}_2F_1(\ell+k+\frac{\nu-1}{2},k;\ell+2k;1-\frac{x^2}{y^2})\,g(y)\,dy.$$

PROOF. If $\ell = 0$, then (2.9) reduces to (2.4). If $\ell \geq 1$, then application of $(d^2/dx^2 + \nu x^{-1}\, d/dx)^k$ to the right hand side of (2.9) yields the right hand side of (2.2). □

The integrand in (2.9) is found by combination of (2.2) and (2.4). This provides the following integral:

$$\int_x^z z\left(\frac{z^2-y^2}{2}\right)^{\ell-1}\left(\frac{y^2-x^2}{2y}\right)^{2k-1} {}_2F_1(k+\frac{\nu-1}{2},k;2k;1-\frac{x^2}{y^2})\,dy$$

$$= \frac{\Gamma(\ell)\Gamma(2k)}{\Gamma(\ell+2k)}\left(\frac{z^2-x^2}{2}\right)^{\ell+2k-1} z^{1-2k}\, {}_2F_1(\ell+k+\frac{\nu-1}{2},k;\ell+2k;1-\frac{x^2}{z^2}),$$

which is a special case of:

$$(2.10)\qquad y^{c+\mu-1}(1-y)^{a-c}\,{}_2F_1(a,b+\mu;c+\mu;y) =$$

$$= \frac{\Gamma(c+\mu)}{\Gamma(c)\Gamma(\mu)}\int_0^y (y-x)^{\mu-1}(1-x)^{a-c-\mu}x^{c-1}\,{}_2F_1(a,b;c;x)\,dx.$$

The above formula has been derived by ASKEY & FITCH [1, th. 2.3] from Bateman's integral

$$(2.11)\qquad {}_2F_1(a,b;c+\mu;x) = \frac{\Gamma(c+\mu)}{\Gamma(c)\Gamma(\mu)}\int_0^1 y^{c-1}(1-y)^{\mu-1}\,{}_2F_1(a,b;c;xy)\,dy.$$

To conclude this section, we summarize the results for the analogous initial value problems considered for $x \geq 1$. Their proofs are completely comparable to those for $0 < x \leq 1$. For theorems 2.5 and 2.6 this is clear from the second expression given for the solution.

LEMMA 2.4. *Let* $g \in C([1,\infty))$, *then the unique solution of*

$$\begin{cases} \left(\frac{1}{x}\frac{d}{dx}\right)^{\ell} f(x) = g(x), & x \ge 1, \\ f^{(i)}(1) = 0, & i = 0,1,2,\dots,\ell-1, \\ f \in C^{\ell}([1,\infty)), \end{cases}$$

is given by

$$f(x) = \frac{1}{\Gamma(\ell)} \int_1^x y\left(\frac{x^2-y^2}{2}\right)^{\ell-1} g(y)\,dy.$$

THEOREM 2.5. *Let* $g \in C([1,\infty))$, *then the unique solution of*

$$\begin{cases} \left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^{k} f(x) = g(x), & x \ge 1,\ \nu \in \mathbb{C}, \\ f^{(i)}(1) = 0, & i = 0,1,\dots,2k-1, \\ f \in C^{2k}([1,\infty)), \end{cases}$$

is given by

$$f(x) = \frac{1}{\Gamma(2k)} \int_1^x \left(\frac{y}{x}\right)^{\nu} \left(\frac{x^2-y^2}{2x}\right)^{2k-1} {}_2F_1\left(k+\frac{\nu-1}{2},k;2k;1-\frac{y^2}{x^2}\right) g(y)\,dy.$$

The right hand side of the solution can also be written as

$$\frac{1}{\Gamma(2k)} \int_1^x \left(\frac{x^2-y^2}{2y}\right)^{2k-1} {}_2F_1\left(k+\frac{\nu-1}{2},k;2k;1-\frac{x^2}{y^2}\right) g(y)\,dy,$$

by the use of the transformation

$$(2.12) \qquad {}_2F_1(a,b;c;z) = (1-z)^{-a}\, {}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right),$$

see ERDÉLYI et al. [5]. We prefer the form of the ${}_2F_1$ as given in the theorem, because in that form its argument takes values in (0,1). In special cases theorem 2.5 can also by obtained by solving the equivalent problem for the wave operator in $\mathbb{R}^{\nu+1}$, analogous to our description after the proof of theorem 2.2. In contrast to the previous case we now have to consider distributions on $\mathbb{R}^{\nu+1}$ with their support bounded to the left by the right hand sheet of the hyperbola $x_0^2 - x_1^2 - \dots - x_\nu^2 = 1$.

Combination of lemma 2.4 and theorem 2.5 leads to

THEOREM 2.6. *Let* $g \in C([1,\infty))$, *then the unique solution* f *of*

$$\left(\frac{1}{x}\frac{d}{dx}\right)^{\ell}\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^{k} f(x) = g(x), \qquad x \geq 1,$$

$$f^{(i)}(1) = 0, \qquad i = 0,1,2,\dots,\ell+2k-1,$$

$$f \in C^{\ell+2k}([1,\infty)),$$

is given by

$$f(x) = \frac{1}{\Gamma(\ell+2k)} \int_1^x \left(\frac{x^2-y^2}{2x}\right)^{\ell+2k-1} \left(\frac{y}{x}\right)^{\nu+\ell} y^{\ell}$$

$$_2F_1\left(\ell+k+\frac{\nu-1}{2}, \ell+k; \ell+2k; 1-\frac{y^2}{x^2}\right) g(y)\,dy.$$

The last expression is equal to:

$$\frac{1}{\Gamma(\ell+2k)} \int_1^x \left(\frac{x^2-y^2}{2}\right)^{\ell+2k-1} y^{1-2k} \, {}_2F_1\left(\ell+k+\frac{\nu-1}{2}, k; \ell+2k; 1-\frac{x^2}{y^2}\right) g(y)\,dy,$$

which can be verified by using (2.12).

In this section we presented the solutions of a number of initial value problems with initial values given for $x = 1$. This value $x = 1$ is arbitrary and the results will be the same if we give initial values for $x = M$, $0 < M < \infty$ and if we replace $\int_x^1$ by $\int_x^M$ and $\int_1^x$ by $\int_M^x$. This can readily be shown by the substitution of $x = x'/M$. In section 8 we will consider the I.V.P. in distributional sense and for arbitrary starting points. In that section we will give the solutions of the IVP for the adjoint operators

$$\frac{d}{dx}\frac{1}{x} \quad \text{and} \quad \frac{d^2}{dx^2} - \frac{d}{dx}\frac{\nu}{x}.$$

3. THE FRACTIONAL INTEGRAL OPERATOR $I_\nu^{\mu,\lambda}$

In section 2 we found the solutions of some I.V.P.'s. We will use the solution of the I.V.P. in theorem 2.3 to define a fractional integral operator.

DEFINITION 3.1. Let $f \in C((0,1])$. Let $\mathrm{Re}(\lambda+\mu) > 0$. then

$$I_\nu^{\mu,\lambda} f(x) := \frac{1}{\Gamma(\lambda+\mu)} \int_x^1 \left(\frac{y^2-x^2}{2}\right)^{\lambda+\mu-1} y^{1-\mu} \; {}_2F_1\left(\frac{2\lambda+\mu+\nu-1}{2}, \frac{\mu}{2}; \lambda+\mu; 1-\frac{x^2}{y^2}\right) f(y)\,dy. \tag{3.1}$$

If $\mu = 0$ then $I_\nu^{\mu,\lambda}$ does not depend on ν. We will use the notation $I^{0,\lambda}$ in that case. Note that $I_\nu^{\mu,\lambda} f$ is continuous on $(0,1]$ and that $I_\nu^{\mu,\lambda} f(1) = 0$. Also note that for each $f \in C((0,1])$ and $x \in (0,1]$, $I_\nu^{\mu,\lambda} f(x)$ depends analytically on λ, μ and ν for $\mathrm{Re}(\lambda+\mu) > 0$. We introduce

$$K_\nu^{\mu,\lambda}(x,y) := \frac{1}{\Gamma(\lambda+\mu)} \left(\frac{y^2-x^2}{2}\right)^{\lambda+\mu-1} y^{1-\mu} \; {}_2F_1\left(\frac{2\lambda+\mu+\nu-1}{2}, \frac{\mu}{2}; \lambda+\mu; 1-\frac{x^2}{y^2}\right), \quad 0 < x < y < 1. \tag{3.2}$$

Thus we have for $\mathrm{Re}(\lambda+\mu) > 0$:

$$I_\nu^{\mu,\lambda} f(x) = \int_x^1 K_\nu^{\mu,\lambda}(x,y) f(y)\,dy.$$

From the relation:

$${}_2F_1(a,b;c;z) = 1 + \frac{ab}{c}\, z\, {}_3F_2(a+1,b+1,1;c+1,2;z)$$

it follows that:

$$I_\nu^{\mu,\lambda} f(x) = \frac{1}{\Gamma(\lambda+\mu)} \int_x^1 \left(\frac{y^2-x^2}{2}\right)^{\lambda+\mu-1} y^{1-\mu} f(y)\,dy$$

$$(3.3)\qquad + \frac{\mu(2\lambda+\mu+\nu-1)}{2\Gamma(\lambda+\mu+1)} \int_x^1 \left(\frac{y^2-x^2}{2}\right)^{\lambda+\mu} y^{-1-\mu} \cdot {}_3F_2\left(\frac{2\lambda+\mu+\nu+1}{2}, \frac{\mu+2}{2}, 1; \lambda+\mu+1, 2; 1-\frac{x^2}{y^2}\right) f(y)\,dy.$$

If $(\lambda+\mu) \downarrow 0$ then the first term of the right hand side of (3.3) (being an ordinary Riemann-Liouville integral) approaches $x^{-\mu}f(x)$ for $0 < x < 1$ and equals zero for $x = 1$. If $\mu = 0$, then the second term in (3.3) is zero. Hence, the following lemma holds.

LEMMA 3.2. *For* $\mu,\lambda \in \mathbb{R}$, $f \in C((0,1])$, $f(1) = 0$,

$$\lim_{\substack{(\mu,\lambda)\to(0,0)\\(\mu+\lambda)>0}} I_\nu^{\mu,\lambda} f(x) = f(x).$$

For $\frac{\mu}{2}$ and λ being nonnegative integers, $I_\nu^{\mu,\lambda}\, g$ is the solution of the I.V.P.

$$\begin{cases} \left(-\frac{1}{x}\frac{d}{dx}\right)^{\lambda}\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^{\frac{\mu}{2}} f(x) = g(x), & 0 < x \le 1, \\ f^{(i)}(1) = 0, & i = 0,1,2,\dots,\lambda+\mu-1. \end{cases}$$

In order to find the composition properties of $I_\nu^{\mu,\lambda}$ we notice that by (1.10):

$$\left(-\frac{1}{x}\frac{d}{dx}\right)^{\lambda_1}\left(\frac{d^2}{dx^2} + \frac{\nu+2\lambda_2}{x}\frac{d}{dx}\right)^{\frac{\mu_1}{2}}\left(-\frac{1}{x}\frac{d}{dx}\right)^{\lambda_2}\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^{\frac{\mu_2}{2}} =$$

$$\left(-\frac{1}{x}\frac{d}{dx}\right)^{\lambda_1+\lambda_2}\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^{\frac{\mu_1+\mu_2}{2}}.$$

Thus we have for $\mu_1/2, \mu_2/2, \lambda_1, \lambda_2 \in \mathbb{N}$:

$$I_\nu^{\mu_2,\lambda_2}\, I_{\mu+2\lambda_2}^{\mu_1,\lambda_1} f(x) = I_\nu^{\mu_1+\mu_2,\lambda_1+\lambda_2} f(x).$$

The following lemma states that this expression also holds for those μ_i, λ_i, $i = 1,2$ for which $I_\nu^{\mu_i,\lambda_i}$ is defined.

LEMMA 3.3. *Let* $\nu,\mu_i,\lambda_i \in \mathbb{C}$, $\mathrm{Re}(\lambda_i+\mu_i) > 0$, $i = 1,2$, *and let* $f \in C((0,1])$. *Then*

$$I_\nu^{\mu_2,\lambda_2}\, I_{\nu+2\lambda_2}^{\mu_1,\lambda_1}\, f(x) = I_\nu^{\mu_1+\mu_2,\lambda_1+\lambda_2}\, f(x). \tag{3.4}$$

Note that the operators $\mu \mapsto I_\nu^{\mu,0}$ form a semigroup.

PROOF.

$$I_\nu^{\mu_2,\lambda_2}\, I_{\nu+2\lambda_2}^{\mu_1,\lambda_1}\, f(x) =$$

$$\int_x^1 K_\nu^{\mu_2,\lambda_2}(x,y)\Big[\int_y^1 K_{\nu+2\lambda_2}^{\mu_1,\lambda_1}(y,z)f(z)dx\Big]dy =$$

$$\int_x^1 \Big[\int_x^z K_\nu^{\mu_2,\lambda_2}(x,y)K_{\nu+2\lambda_2}^{\mu_1,\lambda_1}(y,z)dy\Big]f(z)dz.$$

The expression between brackets equals $K_\nu^{\mu_1+\mu_2,\lambda_1+\lambda_2}(x,z)$ by Erdélyi's integral formula (1.11) with $a = \frac{1}{2}(\nu+2\lambda_1+2\lambda_2+\mu_1+\mu_2-1)$, $b = \frac{1}{2}(\mu_1+\mu_2)$, $c = \lambda_1 + \lambda_2 + \mu_1 + \mu_2$, $r = \frac{1}{2}(\nu+2\lambda_1+2\lambda_2+2\mu_1+\mu_2-1)$, $s = \lambda_1 + \mu_1$, $z \to 1 - \frac{x^2}{z^2}$ and $w \to \frac{z^2-y^2}{z^2-x^2}$. □

In order to extend the definition of $I_\nu^{\mu,\lambda}$ to nonpositive values of μ and λ we first state the following lemmata.

LEMMA 3.4. *Let* $f \in C^1((0,1])$, $f(1) = 0$, *then*

$$\text{(i)} \qquad I^{0,1}\Big(-\frac{1}{x}\frac{d}{dx}\Big)f(x) = f(x), \qquad 0 < x \le 1,$$

and

$$\text{(ii)} \qquad \Big(-\frac{1}{x}\frac{d}{dx}\Big)I^{0,1}f(x) = f(x), \qquad 0 < x \le 1.$$

LEMMA 3.5. *Let* $f \in C^2((0,1])$, $f(1) = f'(1) = 0$, *then*

(i) $$I_\nu^{2,0}\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)f(x) = f(x), \qquad 0 < x \le 1,$$

and

(ii) $$\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)I^{2,0}f(x) = f(x), \qquad 0 < x \le 1.$$

PROOF. The two lemmata are direct corollaries of theorem 2.3. □

REMARK 3.6. In the second part of both lemma 3.3 and 3.4 it is sufficient to suppose that $f \in C((0,1])$.

Now we can extend definition 3.1 to the other values of $\mu,\lambda \in \mathbb{C}$, such that lemma 3.3 holds for all $\mu,\lambda \in \mathbb{C}$.

DEFINITION 3.7. Let $f \in C((0,1])$, $f(1) = 0$, then

$$I^{0,0}f(x) := f(x).$$

Let $f \in C^n((0,1])$, $f^i(1) = 0$, $i = 0,1,\ldots,n-1$, let $\mathbb{R}\mathrm{e}\,(\lambda+\mu) > -n$, then

$$I_\nu^{\mu,\lambda}f(x) := I_\nu^{\mu,n+\lambda}\left(-\frac{1}{x}\frac{d}{dx}\right)^n f(x).$$

This definition does not depend on the choice of n because of lemma 3.4 and (3.4). For $\mathbb{R}\mathrm{e}\,(\lambda+\mu) > -n$ and f as in the second part of definition 3.7 we can also write

(3.5) $$I_\nu^{\mu,\lambda}f(x) = \left(-\frac{1}{x}\frac{d}{dx}\right)^n I_{\nu-2n}^{\mu,n+\lambda}f(x).$$

In order to check that this is equivalent to definition 3.7 note that $f(x) = I^{0,n}g(x)$, where $g(x) = (-x^{-1}\, d/dx)^n f(x)$ and so (3.5) follows from

$$I_\nu^{\mu,\lambda}f(x) = I_\nu^{\mu,\lambda}\, I^{0,n}g(x) = I_\nu^{\mu,\lambda+n}g(x) =$$

$$= \left(-\frac{1}{x}\frac{d}{dx}\right)^n I^{0,n}\, I_\nu^{\mu,\lambda+n}g(x) = \left(-\frac{1}{x}\frac{d}{dx}\right)^n I_{\nu-2n}^{\mu,\lambda+2n}g(x) =$$

$$= \left(-\frac{1}{x}\frac{d}{dx}\right)^n I_{\nu-2n}^{\mu,\lambda+n}f(x).$$

In the same way we prove

$$(3.6)\qquad I^{0,-1}f(x) = (-\frac{1}{x}\frac{d}{dx})f(x),$$

for $f \in C^1((0,1])$, $f(1) = f'(1) = 0$, and

$$(3.7)\qquad I_\nu^{-2,0}f(x) = (\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx})f(x),$$

for $f \in C^2((0,1])$, $f(1) = f'(1) = f''(1) = 0$.

REMARK 3.8. The composition relation (3.4) holds for all $\nu,\mu_i,\lambda_i \in \mathbb{C}$, $i = 1,2$, provided that the function f can be often enough differentiated with suitable conditions in $x = 1$ (cf. definition 3.7).

Let $f \in C^\infty((0,1])$ with $f^{(i)}(1) = 0$, for each $i \in \mathbb{N}$, then

$$I_\nu^{\mu,\lambda}f(x), \qquad 0 < x \leq 1$$

is defined for all $\mu,\lambda \in \mathbb{C}$. From the definition of $I_\nu^{\mu,\lambda}f(x)$ it is clear that this is an analytical function of μ,λ and ν for $\mathrm{Re}(\lambda+\mu) > -n$, where n is an arbitrary natural number. Thus $I_\nu^{\mu,\lambda}f(x)$ is an entire function of μ, λ and ν.

Note that for continuation of $I_\nu^{\mu,\lambda}f(x)$ to negative values of μ and λ, the function f must be in some class $C^n((0,1])$ with appropriate initial conditions for $x = 1$. In section 8 we will extend the definition of $I_\nu^{\mu,\lambda}$.

Using lemma 3.3 and definition 3.7 we have

$$I_\nu^{\mu,\lambda}\, I_{\nu+2\lambda}^{-\mu,-\lambda}f = I_{\nu+2\lambda}^{-\mu,-\lambda}\, I_\nu^{\mu,\lambda}f = f,$$

for functions f which satisfy the appropriate conditions. Thus $I_{\nu+2\lambda}^{-\mu,-\lambda}$ is the inverse operator to $I_\nu^{\mu,\lambda}$. In section 5 it will be shown how $I_\nu^{\mu,\lambda}$ and its inverse $I_{\nu+2\lambda}^{-\mu,-\lambda}$ lead to pairs of integral formulas.

We conclude this section with some qualitative aspects of the integral operator $I_\nu^{\mu,\lambda}$.

LEMMA 3.9. *The kernel* $K_\nu^{\mu,\lambda}(x,y)$ *is nonnegative for* $0 < x \le y < 1$

(i) *if* $\nu \ge 1$ *and* $\mu \ge 0$, $\lambda+\mu > 0$ *and* $2\lambda+\mu \ge 1-\nu$,

(ii) *if* $\nu \le 1$ *and* $\mu \ge \nu-1$, $\lambda+\mu > 0$ *and* $2\lambda+\mu \ge 0$.

PROOF. The lemma is a consequence of $\frac{1}{\Gamma(c)}\, {}_2F_1(a,b;c;x) \ge 0$ if $a \ge 0$, $b \ge 0$, $c > 0$ and $0 \le x < 1$, which follows from the power series expansion (1.1). The second part of the lemma is obtained by using

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\, {}_2F_1(c-a,c-b;c;z), $$

cf. ERDÉLYI et al. [5]. □

In order to analyse the behaviour of $I_\nu^{\mu,\lambda}f(x)$ for $x \uparrow 1$ we rewrite this function by means fo the substitution $s = \frac{1-y^2}{1-x^2}$. Then (3.1) yields:

$$ (3.8)\qquad I_\nu^{\mu,\lambda}f(x) = \frac{1}{\Gamma(\mu+\lambda)} \left(\frac{1-x^2}{2}\right)^{\lambda+\mu} \int_0^1 (1-s)^{\lambda+\mu-1}(1-s(1-x^2))^{-\frac{\mu}{2}} $$

$$ {}_2F_1\left(\frac{2\lambda+\mu+\nu-1}{2},\frac{\mu}{2};\lambda+\mu;\frac{(1-x^2)(1-s)}{1-s(1-x^2)}\right) f(\sqrt{1-s(1-x^2)}\,)\,ds. $$

COROLLARY 3.10. *Let* μ, ν, λ *satisfy the conditions of lemma* 3.9. *Let*

$$ f(x) = (1-x^2)^\alpha f^*(x), $$

where $f^* \in C((0,1])$, $f^*(1) \ne 0$, $\alpha > -1$, *then*

$$ I_\nu^{\mu,\lambda}f(x) = (1-x^2)^{\alpha+\lambda+\mu}h(x), $$

where $h \in C((0,1])$, $h(1) \ne 0$.

PROOF. The corollary clearly follows from (3.8) and lemma 3.9. □

Until now we considered the solutions of the I.V.P. in theorem 2.3 only in the interval $0 < x \le 1$. Let $f \in C([0,1])$. Taking $x \downarrow 0$ in (3.1) we see that a sufficient condition for boundedness of $I_\nu^{\mu,\lambda}f(x)$ on $[0,1]$ is given by

$$ {}_2F_1\left(\frac{2\lambda+\mu+\nu-1}{2},\frac{\mu}{2};\lambda+\mu;1\right) < \infty. $$

This results in the condition

$$ \mathrm{Re}\ \nu < 1. $$

Here we supposed $\mathrm{Re}\ \mu > 0$, $\mathrm{Re}\,(\lambda+\frac{\mu}{2}) > 0$. If $\mu = 0$ then the fractional integral does not depend on ν and it converges for $x \downarrow 0$.

4. ERDÉLYI'S INTEGRAL FORMULA

In this section we will show that the formula (1.11) of Erdélyi is intimately connected with the fractional integral operator $I_\nu^{\mu,\lambda}$ introduced in section 3. We have already seen in the proof of lemma 3.3 that the composition property (3.4) results in the integral

$$ \int_x^z K_\nu^{\mu_2,\lambda_2}(x,y)K_{\nu+2\lambda_2}^{\mu_1,\lambda_1}(y,z)\,dy = K_\nu^{\mu_1+\mu_2,\lambda_1+\lambda_2}(x,z) $$

which is equivalent to (1.11). Now compare (1.9) with the I.V.P. (2.8). For $\ell = \lambda$, $k = \frac{\mu}{2}$, $\nu = 2(a+b-c)+1$, $\mathrm{Re}\,(c) > \mathrm{Re}\,(\lambda+\mu) > 0$, the identity (1.9) is equivalent to

$$ \text{(4.1)} \qquad \frac{(1-x^2)^{c-1}}{2^c\Gamma(c)}\,{}_2F_1(a,b;c;1-x^2) = $$

$$ I^{\mu,\lambda}_{2(a+b-c)+1}\,\frac{(1-x^2)^{c-\lambda-\mu-1}}{2^{c-\lambda-\mu}\Gamma(c-\lambda-\mu)}\,{}_2F_1(a-\tfrac{\mu}{2},b-\tfrac{\mu}{2};c-\lambda-\mu;1-x^2) = $$

$$ \frac{1}{\Gamma(\lambda+\mu)}\int_x^1 \left(\frac{y^2-x^2}{2}\right)^{\lambda+\mu-1} y^{1-\mu}\,{}_2F_1(\lambda+\tfrac{\mu}{2}+a+b-c,\tfrac{\mu}{2};\lambda+\mu;1-\tfrac{x^2}{y^2})\ . $$

$$ \frac{(1-y^2)^{c-\lambda-\mu-1}}{2^{c-\lambda-\mu}\Gamma(c-\lambda-\mu)}\,{}_2F_1(a-\tfrac{\mu}{2},b-\tfrac{\mu}{2};c-\lambda-\mu;1-y^2)\,dy. $$

So (4.1) is proved by theorem 2.3, for $\lambda,\frac{\mu}{2} \in \mathbb{N}$. With the transformation of variables $x^2 = 1-z$ and $y^2 = 1-wz$, (4.1) transforms into (1.11), with values of the parameters:

$$r = a + b - \frac{\mu}{2} \quad \text{and} \quad s = c - \lambda - \mu.$$

So (4.1) with $\mu = 2(a+b-r)$ and $\lambda = c - 2a - 2b + 2r - s$ is equivalent to (1.11), and thus (4.1) holds also for noninteger complex values of $\frac{\mu}{2}$ and λ, such that $\mathrm{Re}\ c > \mathrm{Re}\,(\lambda+\mu) > 0$. We did not prove (4.1) for general values of λ and μ. It is possible, however, to apply Carlson's theorem for analytical continuation with respect to μ and λ and to obtain in this way (4.1) for $\mathrm{Re}\ c > \mathrm{Re}\,(\lambda+\mu) > 0$, and thus (1.11).

5. HYPERGEOMETRIC INTEGRAL EQUATIONS

The operator $I_{\nu}^{\mu,\lambda}$ together with its inverse $I_{\nu+2\lambda}^{-\mu,-\lambda}$ leads to pairs of integral formulas. In particular, by an application of these operators, a new proof can be given of a theorem of WIMP [16]. Wimp proved this by means of Laplace transformation.

THEOREM 5.1. (WIMP). *Let* n *be an integer,* $n > \mathrm{Re}\,(c) > 0$, *let* $0 < x \le 1$, *let* $F \in C^n((0,1])$ *and* $G \in C((0,1])$, *and let* $F(1) = F'(1) = \ldots = F^{(n-1)}(1) = 0$. *Then either of the statements:*

(i) $$F(x) = \int_x^1 (y-x)^{c-1} {}_2F_1(a,b;c;1-\frac{x}{y})G(y)\,dy,$$

(ii) $$G(x) = \frac{(-1)^n}{\Gamma(c)\Gamma(n-c)} \int_x^1 (y-x)^{n-c-1} {}_2F_1(-a,-b;n-c;1-\frac{y}{x})F^{(n)}(y)\,dy,$$

implies the other.

We will obtain this theorem in the following form.

THEOREM 5.1a. *Let* n *be an integer,* $n > \mathrm{Re}\,(\lambda+\mu) > 0$. *Let* $g \in C((0,1])$ *and* $f \in C^n((0,1])$, *let* $f^{(i)}(1) = 0$, $i = 0,1,\ldots,n-1$. *Then the following two statements are equivalent.*

(i) $$f(x) = \frac{1}{\Gamma(\lambda+\mu)} \int_x^1 \left(\frac{y^2-x^2}{2}\right)^{\lambda+\mu-1} y^{1-\mu} {}_2F_1\left(\frac{2\lambda+\mu+\nu-1}{2},\frac{\mu}{2};\lambda+\mu;1-\frac{x^2}{y^2}\right) g(y)\,dy,$$

(ii) $$g(x) = \frac{1}{\Gamma(n-\lambda-\mu)} \int_x^1 \left(\frac{y^2-x^2}{2}\right)^{n-\lambda-\mu-1} y^{1+\mu} \cdot$$

$$\cdot\ {}_2F_1\left(\frac{2n-\mu+\nu-1}{2},-\frac{\mu}{2};n-\lambda-\mu;1-\frac{x^2}{y^2}\right)\left(-\frac{1}{y}\frac{d}{dy}\right)^n f(y)\,dy.$$

PROOF. The first statement is $f(x) = I_\nu^{\mu,\lambda} g(x)$, while the second one is

$$g(x) = \{I_\nu^{\mu,\lambda}\}^{-1} f(x) = I_{\nu+2\lambda}^{-\mu,-\lambda} f(x) = I_{\nu+2\lambda}^{-\mu,-\lambda+n}\left(-\frac{1}{x}\frac{d}{dx}\right)^n f(x).$$

The conditions on f and g are compatible with definition 3.7. □

Theorem 5.1a implies theorem 5.1. This becomes clear from the following substitutions:

$y^{-\mu} g(y) \to G(y^2)$,

$\Gamma(\mu+\lambda) 2^{\mu+\lambda} f(x) \to F(x^2)$,

$x^2, y^2 \to x, y$,

$\mu \to 2a$,

$2\lambda + \mu + \nu - 1 \to 2b$,

$\lambda + \mu \to c$,

and by using (2.12).

Another expression for the inverse $I_{\nu+2\lambda}^{-\mu,-\lambda}$ of $I_\nu^{\mu,\lambda}$ is given by

(5.1) $$I_{\nu+2\lambda}^{-\mu,-\lambda} = I_{\nu+2\lambda}^{2m-\mu,n-\lambda}\left(-\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{d^2}{dx^2}+\frac{\nu}{n}\frac{d}{dx}\right)^m, \qquad \mathrm{Re}(\mu+\lambda) < 2m+n.$$

This can be easily derived from definition 3.7. The condition on f(x) is $f \in C^{n+2m}$, and $f^{(i)}(1) = 0$, $i = 0,1,\ldots,n+2m-1$. Hence (5.1) leads to the following generalization of theorem 5.1a.

THEOREM 5.2. *Let* $n,m \in \mathbb{N}$, $n+2m > \mathrm{Re}(\lambda+\mu) > 0$. *let* $g \in C((0,1])$, *and let* $f \in C^{2m+n}((0,1])$ *and let* $f^{(i)}(1) = 0$, $i = 0,1,2,\dots,n+2m-1$. *Then either of the statements.*

(i) $$f(x) = \frac{1}{\Gamma(\lambda+\mu)} \int_x^1 \left(\frac{y^2-x^2}{2}\right)^{\lambda+\mu-1} y^{1-\mu} {}_2F_1\left(\frac{2\lambda+\mu+\nu-1}{2}, \frac{\mu}{2}; \lambda+\mu; 1-\frac{x^2}{y^2}\right) g(y)\,dy,$$

(ii) $$g(x) = \frac{1}{\Gamma(n+2m-\lambda-\mu)} \int_x^1 \left(\frac{y^2-x^2}{2}\right)^{n+2m-\lambda-\mu-1} y^{1-2m+\mu}\,.$$

$$ {}_2F_1\left(\frac{2n+2m-\mu+\nu-1}{2}, \frac{2m-\mu}{2}; n+2m-\lambda-\mu; 1-\frac{x^2}{y^2}\right).$$

$$\left(-\frac{1}{y}\frac{d}{dy}\right)^n \left(\frac{d^2}{dy^2} + \frac{\nu}{y}\frac{d}{dy}\right)^m f(y)\,dy,$$

implies the other.

Modifying this theorem in about the same way as we did theorem 5.1a, we get

THEOREM 5.2a. *Let* n, m, f *and* g *be as in theorem 5.2. Let* $n+2m > \mathrm{Re}(c) > 0$. *Then either of the statements*

(i) $$F(x) = \frac{1}{\Gamma(c)} \int_x^1 (y-x)^{c-1} y^{-a} {}_2F_1\left(a,b;c;1-\frac{x}{y}\right) G(y)\,dy,$$

(ii) $$G(x) = \frac{(-1)^n}{\Gamma(n+2m-c)} \int_x^1 (y-x)^{n+2m-c-1} y^{-m+a}\,.$$

$$ {}_2F_1\left(m-a, n+m+b-c; n+2m-c; 1-\frac{x}{y}\right).$$

$$\left(\frac{d}{dy}\right)^n \left(y\frac{d^2}{dy^2} + (a+b-c+1)\frac{d}{dy}\right)^m F(y)\,dy,$$

implies the other.

Here the following substitutions were used

$$g(y) \to G(y^2),$$

$$2^c f(x) \to F(x^2),$$

$$x^2, y^2 \to x, y,$$

$$\mu \to 2a,$$

$$2\lambda+\mu+\nu-1 \to 2b,$$

$$\lambda+\mu \to c.$$

6. FURTHER PROPERTIES OF $I_\nu^{\mu,\lambda}$. CONNECTION WITH ERDÉLYI-KOBER OPERATORS

Using the composition property

$$(3.4) \qquad I_\nu^{\mu_2,\lambda_2} I_{\nu+2\lambda_2}^{\mu_1,\lambda_1} = I_\nu^{\mu_1+\mu_2,\lambda_1+\lambda_2}$$

we will split up $I_\nu^{\mu,\lambda}$ into more elementary parts. First we consider the cases for which $I_\nu^{\mu,\lambda}$ can be simplified. A number of these cases is connected with formulas (1.2) and (1.3). An appropriate choice of the parameters yields expressions of $I_\nu^{\mu,\lambda}$ in terms of the Riemann-Liouville integral $I^{0,\lambda}$ and the Erdélyi-Kober integral. The latter is defined by

$$(6.1) \qquad K_{\eta,\alpha} f(x) := \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^\infty (u^2-x^2)^{\alpha-1} u^{-2\alpha-2\eta+1} f(u)\,du.$$

see SNEDDON [13]. If we assume f(x) to be zero for $x \geq 1$, we have

$$(6.2) \qquad K_{\eta,\alpha} f(x) = 2^\alpha x^{2\eta}\, I^{0,\alpha}\, x^{-2\alpha-2\eta},$$

which is clear from (3.1). Considering (3.3), we find for $f \in C((0,1])$, $f(1) = 0$, $\mu \in \mathbb{R}$,

$$(6.3) \qquad I_{\mu+1}^{\mu,-\mu} f(x) = x^{-\mu} f(x).$$

Combination of (6.3), (3.4) and (6.2) yields the following special cases:

(6.4) $$I^{\mu,\lambda}_{1-\mu-2\lambda} = I^{0,\lambda+\mu}\, I^{\mu,-\mu}_{\mu+1} = I^{0,\lambda+\mu} x^{-\mu} = \frac{x^{2\lambda+\mu}}{2^{\lambda+\mu}} K_{-\lambda-\frac{\mu}{2},\lambda+\mu},$$

(6.5) $$I_{\nu}^{\mu,-\frac{\mu}{2}} = I_{\nu}^{\nu-1,-\nu+1}\, I^{0,\frac{\mu}{2}}\, I^{\mu-\nu+1,-\mu+\nu-1}_{\mu-\nu+2} = x^{-\nu+1}\, I^{0,\frac{\mu}{2}}\, x^{-\mu+\nu-1}$$

$$= 2^{-\frac{\mu}{2}} K_{-\frac{\nu-1}{2},\frac{\mu}{2}},$$

(6.6) $$I^{\mu,\lambda}_{\mu+1} = I^{\mu,-\mu}_{\mu+1}\, I^{0,\lambda+\mu} = x^{-\mu}\, I^{0,\lambda+\mu}.$$

Another special case of $I_{\nu}^{\mu,\lambda}$ is suggested by the fact that the operator $\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}$ equals $\frac{d^2}{dx^2}$ if $\nu = 0$. Thus $\lambda = \nu = 0$ must result in an ordinary Riemann-Liouville integral for $I_0^{\mu,0}$. Indeed apply

(6.7) $${}_2F_1(a-\tfrac{1}{2},a;2a;z) = \{\tfrac{1}{2}+\tfrac{1}{2}(1-z)\}^{1-2a}$$

in $K_{\nu}^{\mu,\lambda}(x,y)$ (cf. ERDÉLYI et al. [5]). We distinguish two different cases:
(i) $a = \frac{\mu}{2}$, $\lambda = 0$, $\nu = 0$. Then

(6.8) $$I_0^{\mu,0} f(x) = \frac{1}{\Gamma(\mu)} \int_x^1 (y-x)^{\mu-1} f(y)\,dy,$$

which is in accordance with the expected Riemann-Liouville integral.
(ii) $a = \frac{\mu+1}{2}$, $\lambda = 1$, $\nu = 0$. Then

(6.9) $$I_0^{\mu,1} f(x) = \frac{1}{\Gamma(\mu+1)} \int_x^1 (y-x)^{\mu} y f(y)\,dy.$$

This outcome could be expected from (6.8) because substitution of $f(x) = (-\frac{1}{x}\frac{d}{dx})g(x)$ in (6.9) and partial integration leads to (6.8) again.

Originally we defined the operator $I_{\nu}^{\mu,\lambda}$ by stating

$$I_{\nu}^{\mu,\lambda} = I_{\nu}^{\mu,0}\, I^{0,\lambda},$$

(see the discussion after the proof of theorem 2.3). Now we can ask the question: is it possible to insert a certain power of x between $I_\nu^{\mu,0}$ and $I^{0,\lambda}$ without affecting the description of the kernel as a product of a hypergeometric function and powers of elementary functions of x and y? From (6.3) and (3.4) we derive the case

$$I_\nu^{\mu,0}\, x^{1-\nu}\, I^{0,\lambda} = I_\nu^{\mu+\nu-1,\lambda-\nu+1}. \tag{6.10}$$

From the calculation after the proof of theorem 2.3 and from formula (2.10) used there we only find that x^0 or $x^{1-\nu}$ can be inserted in that place. The special role of $x^{1-\nu}$ is clarified by the following observation:

Let $\mu = 2k$, $\lambda = \ell$, then

$$f(x) = I_\nu^{2k,0}\, x^{1-\nu}\, I^{0,\ell} g(x) \tag{6.11}$$

is the solution of the I.V.P.

$$\begin{cases} \left(-\frac{1}{x}\frac{d}{dx}\right)^{\ell} x^{\nu-1}\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^k f(x) = g(x), & 0 < x \le 1, \\ f^{(i)}(1) = 0, & i = 0,1,2,\ldots,\ell+2k-1. \end{cases} \tag{6.12}$$

Note

$$\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx} = x^{-\nu}\frac{d}{dx}x^{\nu}\frac{d}{dx}$$

implies that the adjoint operator equals

$$\frac{d^2}{dx^2} - \frac{d}{dx}\frac{\nu}{x} = \frac{d}{dx}x^{\nu}\frac{d}{dx}x^{-\nu} = x^{\nu}\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)x^{-\nu}.$$

Hence the operator in (6.12) can be written in the form

$$(-1)^{\ell} x^{-1}\left[\left(\frac{d}{dx}x^{-1}\right)^{\ell}\left(\frac{d^2}{dx^2} - \frac{d}{dx}\frac{\nu}{x}\right)^k\right]x^{\nu} \tag{6.13}$$

and the operator in brackets is adjoint to

$$\left(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}\right)^k \left(-\frac{1}{x}\frac{d}{dx}\right)^{\ell}.$$

Thus a natural completement to the initial value problems of section 2 is given by the corresponding problems concerning the adjoint operators. By the use of the equality of (6.13) and the differential operator in (6.12) it is not difficult to give an expression for the solution of I.V.P.'s with the adjoint operators in terms of solutions of the corresponding I.V.P.'s. In section 8 we will derive these solutions in an explicit way by using the method of fractional integration by parts. There the property is exploited that these operators are adjoint to the operators in section 2.

Let us write $I_\nu^{\mu,\lambda}$ in terms of combinations of $x^{-\mu} = I_{\mu+1}^{\mu,-\mu}$, $I^{0,\lambda}$ and $I_0^{\mu,0}$. The following results can be checked from (3.4):

$$I_\nu^{\mu,\lambda} = I^{0,\frac{\mu-\nu+1}{2}}\, x^{-\mu}\, I^{0,\lambda+\frac{\nu+\mu-1}{2}}, \tag{6.14}$$

$$I_\nu^{\mu,\lambda} = I^{0,-\frac{\nu}{2}}\, I_0^{\mu,0}\, I^{0,\frac{\nu}{2}+\lambda}. \tag{6.15}$$

Formula (6.14) results in

$$I_{\nu+2\lambda}^{-\mu,-\lambda} f(x) = I^{0,-\lambda-\frac{\nu+\mu-1}{2}}\, x^{\mu}\, I^{0,-\frac{\mu-\nu+1}{2}} f(x).$$

Using this in the second statement of theorem 5.1a with λ, μ and ν expressed in terms of a, b and c according to theorem 5.1, we find

$$g(x) = I^{0,-b}(x^2)^a\, I^{0,b-c} f(x).$$

This last expression corresponds to the solution given by Love, of the first equation in theorem 5.1 (cf. SRIVASTAVA & BUSCHMAN [14, p. 130]).

An important special case of (6.15) is obtained by taking $\lambda = 0$ and $\mu = 2k$, $k \in \mathbb{N}$. Then (6.15) gives the solution of theorem 2.2 in terms of the solution of

$$\frac{d^{2k}}{dx^{2k}} f^*(x) = g^*(x), \qquad 0 < x \le 1, \tag{6.16}$$

where $f^*(x) = I^{0,\nu/2}f(x)$ and $g^*(x) = I^{0,\nu/2}g(x)$. The solution of (6.16) with the appropriate initial conditions is (cf. (6.8)):

$$f^*(x) = \frac{1}{\Gamma(2k)} \int_x^1 (y-x)^{2k-1} g^*(y)\,dy.$$

The fact that the fractional integral operator $I^{0,\nu/2}$ transforms $\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx}$ into $\frac{d^2}{dx^2}$ can be found in various places. We mention the papers by ERDÉLYI [2] and LIONS [8]. Both authors considered initial value problems starting at $x = 0$. Lions considered the partial differential equation

$$\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} - \frac{\nu}{x}\frac{\partial}{\partial x}.$$

7. DISTRIBUTIONS

This section contains a short introduction to the classes of test functions and distributions which are needed in section 8. As references we mention DE JAGER [6] and SCHWARTZ [12].

Let Ω be an open interval in $\mathbb{R}$. We will distinguish the following topological vector spaces of test functions on Ω.

<u>DEFINITION 7.1.</u>

(i) The space $\mathcal{E}(\Omega)$ is defined to be $C^\infty(\Omega)$ endowed with the topology of a Fréchet space such that

$$\lim_{n\to\infty} f_n = 0 \quad \text{iff}$$

$$(7.1) \qquad \lim_{n\to\infty}\{\sup_{x\in K} |f_n^{(i)}(x)|\} = 0$$

for any compact subset $K \subset \Omega$ and any $i \in \mathbb{N}$.

(ii) Let K be some closed subset of Ω, then

$$\mathcal{D}_K(\Omega) := \{f \in \mathcal{E}(\Omega) \mid \text{supp}(f) \subset K\}$$

is a closed subspace of $\mathcal{E}(\Omega)$. Hence, with the inherited topology, $\mathcal{D}_K(\Omega)$ is a Fréchet space. In particular, if K is compact, the convergence on $\mathcal{D}_K(\Omega)$ is given by (7.1) with K fixed.

(iii) $$\mathcal{D}(\Omega) = C_c^\infty(\Omega) = \{f \in \mathcal{E}(\Omega) \mid \mathrm{supp}(f) \subset K \text{ for some compact } K \subset \Omega\}.$$

Considered as a topological vector space, $\mathcal{D}(\Omega)$ is the inductive limit of the spaces $\mathcal{D}_K(\Omega)$ with $K \subset \Omega$ and K compact. Thus the convergence in $\mathcal{D}(\Omega)$ is given by

$$\lim_{n\to\infty} f_n = 0 \quad \text{iff}$$

there exists a compact subset K of Ω such that $\mathrm{supp}(f_n) \subset K$, $n = 0,1,2,\ldots$, and such that (7.1) holds for all $i \in \mathbb{N}$.

(iv) $$\mathcal{D}_+((0,\infty)) = \{f \in \mathcal{E}((0,\infty)) \mid \mathrm{supp}(f) \subset [m,\infty) \text{ for some } m \in (0,\infty)\}.$$

If considered as a topological vector space, $\mathcal{D}_+((0,\infty))$ is the inductive limit of the spaces $\mathcal{D}_{[c,\infty)}((0,\infty))$, $c > 0$. In $\mathcal{D}_+((0,\infty))$

$$\lim_{n\to\infty} f_n = 0 \quad \text{iff } \mathrm{supp}(f_n) \subset [m,\infty)$$

for some fixed $m \in (0,\infty)$ and (7.1) holds for any compact subset $K \subset [m,\infty)$ and any $i \in \mathbb{N}$.

(v) Similarly we have

$$\mathcal{D}_-((0,\infty)) = \{f \in \mathcal{E}((0,\infty)) \mid \mathrm{supp}(f) \subset (0,M] \text{ for some } M \in (0,\infty)\}.$$

as the inductive limit of spaces $\mathcal{D}_{(0,c]}(0,\infty)$. Hence,

$$\lim_{n\to\infty} f_n = 0 \quad \text{iff } \mathrm{supp}(f_n) \subset (0,M]$$

for some fixed $M \in (0,\infty)$ and (7.1) holds for any compact subset $K \subset (0,M]$ and any $i \in \mathbb{N}$.

All these spaces can be found in SCHWARTZ [12, Ch. III § 7, I § 2, III § 1 and VI § 5]. For "inductive limit" see TRÈVES [15, Ch. 13]. A linear functional on a Fréchet space or on an inductive limit of Fréchet spaces is continuous if and only if it is sequentially continuous (for inductive limits of Fréchet spaces this follows from the corollary on p. 128 in TRÈVES [15]. In particular, this applies to the functionals on the spaces in definition 7.1. The duals of $\mathcal{E}$ and $\mathcal{D}$ (i.e. the spaces of continuous linear functionals on $\mathcal{E}$ and $\mathcal{D}$) will be denoted by $\mathcal{E}'$ and $\mathcal{D}'$, respectively. In the remainder of this section we will take $\Omega = (0,\infty)$.

DEFINITION 7.2.

(i) $\qquad \mathcal{D}'_-(\Omega) = \{\phi \in \mathcal{D}'(\Omega) \mid \operatorname{supp}(\phi) \subset (0,M] \text{ for some } M \in (0,\infty)\},$

(ii) $\qquad \mathcal{D}'_+(\Omega) = \{\phi \in \mathcal{D}'(\Omega) \mid \operatorname{supp}(\phi) \subset [m,\infty) \text{ for some } m \in (0,\infty)\}.$

We will state four theorems concerning distributions in $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{D}'_-$ and $\mathcal{D}'_+$. Theorem 7.5 will be proved. The proofs of the theorems 7.4 and 7.6 are similar to that of theorem 7.5. For the proof of theorem 7.3 see SCHWARTZ [12, Ch. III § 6].

THEOREM 7.3. *For any distribution* $\phi \in \mathcal{D}$ *and any compact set* $K \subset \Omega$ *there exist some continuous function* ψ, *and an* $i \in \mathbb{N}$, *such that for all test functions* $f \in \mathcal{D}$ *with* $\operatorname{supp}(f) \subset K$, *the following relation holds:*

$$(\phi,f) = \left(\frac{d^i\psi}{dx^i}, f\right) = (-1)^i \int_K \psi(x)\frac{d^i f}{dx^i}(x)\,dx.$$

THEOREM 7.4.

$$\mathcal{E}'(\Omega) = \{\phi \in \mathcal{D}'(\Omega) \mid \operatorname{supp}(\phi) \text{ is compact in } \Omega\}.$$

THEOREM 7.5. *The dual of* $\mathcal{D}_-(\Omega)$ *is equal to* $\mathcal{D}'_+(\Omega)$.

THEOREM 7.6. *The dual of* $\mathcal{D}_+(\Omega)$ *is equal to* $\mathcal{D}'_-(\Omega)$.

PROOF OF THEOREM 7.5. Let us denote the dual of $\mathcal{D}_-(\Omega)$ by $\mathcal{F}'$. First assume $\psi \in \mathcal{D}'_+(\Omega)$. Then $\operatorname{supp}(\psi) \subset [m,\infty)$ for some $m \in (0,\infty)$. Let $\rho \in C^\infty(\Omega)$ such that

$$\rho(x) = \begin{cases} 0, & x \in (0, \frac{1}{3} m], \\ 1, & x \in [\frac{2}{3} m, \infty), \end{cases}$$

then for each function $g \in \mathcal{D}(\Omega)$, we have

$$(\psi, g) = (\rho\psi, g) = (\psi, \rho g).$$

Let $f \in \mathcal{D}_-(\Omega)$, then obviously $\rho f \in \mathcal{D}(\Omega)$, and we consider the linear functional ϕ on $\mathcal{D}_-(\Omega)$ given by

$$(\phi, f) := (\psi, \rho f).$$

Let (f_n) be a sequence in $\mathcal{D}_-(\Omega)$, which converges to zero, and let $g_n = \rho f_n$, then $\text{supp}(f_n) \subset (0,p] \subset (0,\infty)$ and thus $\text{supp}(g_n) = \text{supp}(\rho f_n) \subset [\frac{1}{3} m, p]$. Hence $g_n \to 0$ in $\mathcal{D}(\Omega)$, and

$$\lim_{n\to\infty} (\phi, f_n) = \lim_{n\to\infty} (\psi, \rho f_n) = \lim_{n\to\infty} (\psi, g_n) = 0.$$

Hence each $\psi \in \mathcal{D}'_+(\Omega)$ extends to a continuous functional on $\mathcal{D}_-(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $\mathcal{D}_-(\Omega)$ this extension is unique.

Conversely, let $\phi \in F'$. Then $\phi \in \mathcal{D}'(\Omega)$, because $g \in \mathcal{D}(\Omega)$ implies $g \in \mathcal{D}_-(\Omega)$ and $g_n \to 0$ in $\mathcal{D}(\Omega)$ implies $g_n \to 0$ in $\mathcal{D}_-(\Omega)$. Now the support of ϕ may not contain zero, since otherwise we could find a sequence (f_n) in $\mathcal{D}_-(\Omega)$ such that

$$\begin{cases} f_n(x) = 0 & \text{if } x \geq \frac{1}{n}, \\ (\phi, f_n) = 1. \end{cases}$$

Clearly $f_n \to 0$ in $\mathcal{D}_-(\Omega)$, which implies $(\phi, f_n) \to 0$, $(\phi \in F')$; a contradiction. Hence, $F' \subset \mathcal{D}'_+(\Omega)$. $\square$

8. THE ACTION OF $I_\nu^{\mu,\lambda}$ WITH RESPECT TO DISTRIBUTIONS

Let $\Omega = (0,\infty)$. We introduce the notation $C_-^k(\Omega)$, $k = 0,1,2,\ldots$, for the set of functions f which are k-times continuously differentiable on Ω and for which there exists an $M \in (0,\infty)$ such that $f(x) = 0$ if $x > M$. Thus the support of a function in C_-^k is bounded from above. We will write $C_-(\Omega)$ for $C_-^0(\Omega)$. Note that $f \in C_-^k(\Omega)$ implies $f^{(i)}(M) = 0$, $i = 0,\ldots,k$ for M sufficiently large. From the discussion at the end of section 2 and the properties of the function spaces $C_-^k(\Omega)$ the following lemma can be readily derived.

LEMMA 8.1. *Let* $g \in C_-(\Omega)$ *then the unique solution* f *in* $C^{\ell+2k}(\Omega)$ *of the I.V.P.*

$$(8.1)\qquad (-\frac{1}{x}\frac{d}{dx})^{\ell}(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx})^k f = g,$$

is given by

$$(8.2)\qquad f(x) = \frac{1}{\Gamma(\ell+2k)}\int_x^\infty (\frac{y^2-x^2}{2})^{\ell+2k-1} y^{1-2k}\cdot$$

$$\cdot {}_2F_1(\ell+k+\frac{\nu-1}{2},k;\ell+2k;1-\frac{x^2}{y^2})g(y)dy.$$

This lemma, which contains theorem 2.3, leads to the following generalization of the definitions of $I_\nu^{\mu,\lambda}$ in section 3. We will use the same symbol $I_\nu^{\mu,\lambda}$ for the fractional integral operator which we define here, assuming it to be clear from the context which definition is meant.

DEFINITION 8.2.

(i) Let $\mathrm{Re}(\lambda+\mu) > 0$ and let $f \in C_-(\Omega)$, then

$$(8.3)\qquad I_\nu^{\mu,\lambda}f(x) := \frac{1}{\Gamma(\lambda+\mu)}\int_x^\infty (\frac{y^2-x^2}{2})^{\lambda+\mu-1} y^{1-\mu}\cdot$$

$$\cdot\ {}_2F_1(\frac{2\lambda+\mu+\nu-1}{2},\frac{\mu}{2};\lambda+\mu;1-\frac{x^2}{y^2})f(y)dy.$$

(ii) Let $f \in C_{-}(\Omega)$, then

$$I^{0,0}f(x) := f(x),$$

and

(iii) $$I^{\mu,-\mu}_{\mu+1}f(x) = x^{-\mu}f(x)$$

(cf. (6.3)).

(iv) Let $f \in C^{n}_{-}(\Omega)$, and let $\mathrm{Re}\,(\lambda+\mu) > -n$, then

$$I^{\mu,\lambda}_{\nu}f(x) := I^{\mu,n+\lambda}_{\nu}\left(-\frac{1}{x}\frac{d}{dx}\right)^{n}f(x).$$

Again

$$K^{\mu,\lambda}_{\nu}(x,y) := \frac{1}{\Gamma(\lambda+\mu)}\left(\frac{y^2-x^2}{2}\right)^{\lambda+\mu-1}y^{1-\mu}\,{}_2F_1\left(\frac{2\lambda+\mu+\nu-1}{2},\frac{\mu}{2};\lambda+\mu;1-\frac{x^2}{y^2}\right), \tag{8.4}$$

$0 < x < y.$

Now the composition property (3.4) holds for all $\mu_i,\lambda_i,\nu \in \mathbb{C}$, $i = 1,2$, provided that the operators act on functions in the appropriate classes.

<u>LEMMA 8.3</u>. *Let* $\mu_i,\lambda_i,\nu \in \mathbb{C}$, $i = 1,2$, *then*

$$I^{\mu_2,\lambda_2}_{\nu}I^{\mu_1,\lambda_1}_{\nu+2\lambda_2} = I^{\mu_1+\mu_2,\lambda_1+\lambda_2}_{\nu}. \tag{8.5}$$

Until now we applied the operators $I^{\mu,\lambda}_{\nu}$ on functions in the classes $C^{n}_{-}(\Omega)$. These operators can be defined for a broader class of functions, namely the distributions in $\mathcal{D}'_{-}(\Omega)$. For this purpose we introduce the adjoint operator $J^{\mu,\lambda}_{\nu}$ of $I^{\mu,\lambda}_{\nu}$. First suppose $\mathrm{Re}\,(\lambda+\mu) > 0$ and $f \in C_{-}(\Omega)$, $\phi \in C_{+}(\Omega)$ (the space $C_{+}(\Omega)$ $(C^{n}_{+}(\Omega))$ consists of functions ϕ in $C(\Omega)$ $(C^{n}(\Omega))$ with $\mathrm{supp}(\phi) \in [m,\infty)$ for some $m > 0$). Then we have

$$\int_0^\infty I_\nu^{\mu,\lambda} f(x)\phi(x)dx = \int_0^\infty [\int_x^\infty K_\nu^{\mu,\lambda}(x,y)f(y)dy]\phi(x)dx =$$

$$\int_0^\infty [\int_0^y K_\nu^{\mu,\lambda}(x,y)\phi(x)dx]f(y)dy = \int_0^\infty f(x)J_\nu^{\mu,\lambda}\phi(x)dx,$$

with

$$(8.6) \qquad J_\nu^{\mu,\lambda}\phi(x) := \int_0^x K_\nu^{\mu,\lambda}(y,x)\phi(y)dy.$$

The operator $J_\nu^{\mu,\lambda}$ can be considered also as a fractional integral operator, since, for $\mu = 2k$ and $\lambda = \ell$, $k,\ell \in \mathbb{N}$, we have the following lemma

LEMMA 8.4. *Let* $\nu \in \mathbb{C}$ *and* $\phi \in C_+(\Omega)$. *Then the unique solution* $\psi \in C_+^{\ell+2k}(\Omega)$ *of the I.V.P.*

$$(8.7) \qquad (\frac{d^2}{dx^2} - \frac{d}{dx}\frac{\nu}{x})^k(\frac{d}{dx}\frac{1}{x})^\ell\psi(x) = \phi(x),$$

is given by

$$(8.8) \qquad \psi = J_\nu^{2k,\ell}\phi.$$

PROOF. Let f and g be as in lemma 8.1. Then

$$\int_0^\infty g(x)J_\nu^{2k,\ell}\phi(x)dx = \int_0^\infty I_\nu^{2k,\ell}g(x)\phi(x)dx =$$

$$\int_0^\infty f(x)[(\frac{d^2}{dx^2} - \frac{d}{dx}\frac{\nu}{x})^k(\frac{d}{dx}\frac{1}{x})^\ell\psi(x)]dx =$$

$$\int_0^\infty [(-\frac{1}{x}\frac{d}{dx})^\ell(\frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx})^k f(x)]\psi(x)dx = \int_0^\infty g(x)\psi(x)dx.$$

Since the function g is in $C_-(\Omega)$, but otherwise arbitrary, this results in $\psi = J_\nu^{2k,\ell}\phi$. □

For $\mathrm{Re}(\lambda+\mu) > 0$ and $\phi \in C_+(\Omega)$, $J_\nu^{\mu,\lambda}$ is defined by (8.6). For other values of λ and μ we define $J_\nu^{\mu,\lambda}$ analogous to $I_\nu^{\mu,\lambda}$:

DEFINITION 8.5.

(i) Let $\phi \in C_+(\Omega)$, then

$$J^{0,0}\phi(x) = \phi(x),$$

and

(ii) $$J_{\mu+1}^{\mu,-\mu}\phi(x) = x^{-\mu}\phi(x).$$

(iii) Let $\phi \in C_+^n(\Omega)$, and let $\mathrm{Re}(\lambda+\mu) > -n$, then

$$J_\nu^{\mu,\lambda}\phi(x) := J_{\nu-2n}^{\mu,n+\lambda}\left(\frac{d}{dx}\frac{1}{x}\right)^n\phi(x).$$

REMARK 8.6. The last expression is obtained by applying

$$I_\nu^{\mu,n+\lambda}\left(-\frac{1}{x}\frac{d}{dx}\right)^n = \left(-\frac{1}{x}\frac{d}{dx}\right)^n I_{\nu-2n}^{\mu,n+\lambda}$$

cf. (3.5).

Lemma 8.3 leads to the following composition property for the operators $J_\nu^{\mu,\lambda}$:

$$J_{\nu+2\lambda_2}^{\mu_1,\lambda_1} J_\nu^{\mu_2,\lambda_2} = J_\nu^{\mu_1+\mu_2,\lambda_1+\lambda_2}. \tag{8.9}$$

THEOREM 8.7. *The operator* $I_\nu^{\mu,\lambda}$ $(J_\nu^{\mu,\lambda})$ *is a continuous mapping from* $\mathcal{D}_-(\Omega)$ $(\mathcal{D}_+(\Omega))$ *onto itself.*

PROOF. We will give the proof for $I_\nu^{\mu,\lambda}$. Let $\mathrm{Re}(\lambda+\mu) > -n$ and $f \in \mathcal{D}_-(\Omega)$. Then $\mathrm{supp}(f) \subset (0,M]$ for some $M \in (0,\infty)$ and

$$\left(-\frac{1}{x}\frac{d}{dx}\right)^m I_\nu^{\mu,\lambda}f(x) = I_{\nu+2m}^{\mu,\lambda+n}\left(-\frac{1}{x}\frac{d}{dx}\right)^{m+n}f(x), \tag{8.10}$$

is continuous and has its support in $(0,M]$, for all $m \in \mathbb{N}$. Thus $I_\nu^{\mu,\lambda}f \in \mathcal{D}_-(\Omega)$. From the definitions of $\mathcal{D}_-(\Omega)$ and $I_\nu^{\mu,\lambda}$ and from (8.10) it is clear that

$I_\nu^{\mu,\lambda}: \mathcal{D}_-(\Omega) \to \mathcal{D}_-(\Omega)$, is sequentially continuous, and thus continuous (compare section 7). The mapping $I_\nu^{\mu,\lambda}$ has an inverse (viz. $I_{\nu+2\lambda}^{-\mu,-\lambda}$) and is therefore surjective. □

DEFINITION 8.8. Let $f \in \mathcal{D}'_-(\Omega)$ and $\phi \in \mathcal{D}_+(\Omega)$, then

$$(I_\nu^{\mu,\lambda} f,\phi) := (f, J_\nu^{\mu,\lambda}\phi).$$

DEFINITION 8.9. Let $f \in \mathcal{D}_-(\Omega)$ and $\phi \in \mathcal{D}'_+(\Omega)$, then

$$(J_\nu^{\mu,\lambda}\phi, f) := (\phi, I_\nu^{\mu,\lambda} f).$$

In view of theorem 8.7 both definitions make sense. From definitions 8.8 and 8.9 and lemmata 8.1 and 8.4 (and its proof) it is clear that lemma 8.1 holds for f and $g \in \mathcal{D}'_-(\Omega)$ and that lemma 8.6 holds for ϕ and $\psi \in \mathcal{D}'_+(\Omega)$.

ACKNOWLEDGEMENT. *I am indebted to Tom Koornwinder for suggesting the subject of this paper and for guidance in the course of writing it.*

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At the end of page 11 there should be added:

In 1 the following asymptotic expression for the normal contact stress $P_s(x)$, in case of a smooth stamp was derived

$$P_s(x) = \frac{P}{\pi\sqrt{1-x^2}} + \frac{p}{2b^2}\left[\frac{\bar{B}_1}{\sqrt{1-x^2}} - \bar{B}_1\sqrt{1-x^2}\right] + O\left(\frac{1}{b^4}\right),$$

$$-1 < x < 1, \qquad (b\to\infty).$$

We have

$$\frac{2(1-\nu)}{\sqrt{\kappa}} = 1 + 2(\tfrac{1}{2}-\nu)^2 + O((\tfrac{1}{2}-\nu)^3), \quad (\nu\to\tfrac{1}{2}).$$

In fact, for $\nu = 0$ we find $\frac{2(1-\nu)}{\sqrt{\kappa}} = 1.1547$.

Moreover, the parts of the contact region where the oscillating term in $(3.19)^1$, $\cos(\beta \log(\frac{1+x}{1-x}))$ differs significantly from 1, lie in very small neighbourhoods of ±1 (cf. [5,p.467]). In fact, for $\nu = 0$, $x = 0.9$ we find $\cos(\beta \log(\frac{1+x}{1-x})) = 0.9670$.
Thus outside these small neighbourhoods and for small $\frac{1}{2} - \nu$, the difference between the normal contact stresses in the cases of a smooth stamp and a rigid one, respectively, is small.